A devil's staircase from rotations and irrationality measures for Liouville numbers*

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Abstract

From Sturmian and Christoffel words we derive a strictly increasing function $\Delta : [0, \infty) \to \mathbb{R}$. This function is continuous at every irrational point, while at rational points, left-continuous but not right-continuous. Moreover, it assumes algebraic integers at rationals, and transcendental numbers at irrationals. We also see that the differentiation of Δ distinguishes some irrationality measures of real numbers.

1 Introduction

For any real number $\alpha \in [0, 1]$, we consider the dynamics of rotation

$$R_\alpha(\rho) = \rho + \alpha \mod 1.$$

Using an alphabet $A = \{a, b\}$ with some integers $0 \leq a < b$, the itinerary $(R_\alpha^n(\rho))_{n \geq 0}$ of some ρ is recorded according to a partition

$$\mathcal{P} = \{[0, 1 - \alpha), [1 - \alpha, 1)\} \quad \text{or} \quad \mathcal{P}' = \{(0, 1 - \alpha], (1 - \alpha, 1]\}.$$

In other words, if $\rho \in [0, 1]$ has traveled over $\mathcal{P}$, then its infinite history $(s_{\alpha,\rho}(n))_{n \geq 0}$ of rotations is determined by the following rule:

$$s_{\alpha,\rho}(n) = \begin{cases} a, & R_\alpha^n(\rho) \in [0, 1 - \alpha); \\ b, & R_\alpha^n(\rho) \in [1 - \alpha, 1). \end{cases}$$

In case that we adopt the partition $\mathcal{P}'$, a sequence $(s'_{\alpha,\rho}(n))_{n \geq 0}$ is also defined in a similar fashion. One can observe that if $A = \{0, 1\}$, then these infinite words are obtained by the formulae

$$s_{\alpha,\rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor, \quad s'_{\alpha,\rho}(n) = \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil,$$

where $\lfloor t \rfloor$ is the largest integer not greater than t, and $\lceil t \rceil$ is the smallest integer not less than t. The infinite words $s_{\alpha,\rho}$, $s'_{\alpha,\rho}$ are called *lower* and *upper*

mechanical words respectively with *slope* α and *intercept* ρ. If the slope α is irrational, then these infinite words are termed *Sturmian words*. For interesting properties of these words, we refer the reader to [18].

While the rotations are 'additive' dynamics, we now turn to 'multiplicative' dynamics. Let $\beta > 1$ be a real number. The *β-transformation* $T_\beta : [0,1] \to [0,1)$ is a map defined by

$$T_\beta(x) = \beta x \mod 1.$$

Then the T_β-orbit of x can be represented according to a partition

$$\mathcal{P} = \{[0, 1/\beta), [1/\beta, 2/\beta), \ldots, [(\lfloor \beta \rfloor - 1)/\beta, \lfloor \beta \rfloor/\beta), [\lfloor \beta \rfloor/\beta, 1)\},$$

that is to say,

$$d_\beta(x) := (x_i)_{i \geq 1} \text{ if and only if } x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor.$$

We call $d_\beta(x)$ the *β-expansion* of x, and the *β-shift* S_β is the closure of $\{d_\beta(x)| x \in [0,1)\}$ in the full shift. If β is an integer, then the β-expansion is nothing but the usual integer base number representation and S_β is the full shift over the alphabet $A = \{0, 1, \ldots, \beta - 1\}$.

The β-expansion launched some study of real numbers. In terms of the morphology of $d_\beta(1)$, Blanchard [6] classified real numbers greater than one into five classes C_1 to C_5. On the other hand, Verger-Gaugry [24] focused on the patterns of the consecutive zeros in $d_\beta(1)$, and defined classes $Q_0^{(1)}$, $Q_0^{(2)}$, $Q_0^{(3)}$, Q_1 and Q_2. These new points of view are quite different from the usual algebraic one. For example, we do not know to which class $\frac{3}{2}$ belongs. See [6] and [24] for the precise definitions of each classes.

In [9], irrational rotations were associated with β-transformations in terms of their itinerary sequences, and this connection naturally extends to rational rotations as explained below. Chi and Kwon showed there that if s is a Sturmian word of irrational slope $\alpha \in (0,1)$ over an alphabet $A = \{a, b\}$ with $0 \leq a < b$, then there exists a unique $\beta \in (b, b+1)$ such that $d_\beta(1) = E(s'_{\alpha,0})$ where E is a morphism $0 \mapsto a$, $1 \mapsto b$. Moreover they also proved that such β's are transcendental numbers, which provide continuum examples of transcendental numbers in the class C_3. This ia a partial answer to the question posed by Blanchard [6]. Recently, a continuum of transcendental numbers in the class C_4 were found in [3].

From different contexts, Sturmian words also yielded transcendental numbers in some numeration systems, e.g., in integer base expansions [11] and in continued fraction expansions [5]. Recently, these two results were quite generalized by Adamczewski and Bugeaud [2, 1].

In the present paper, we will define a function $\Delta : \mathbb{R}^{\geq 0} \to \mathbb{R}$ by $\Delta(\alpha) = \beta$ such that the β-expansion of 1 is a mechanical word of slope α. Then the function enjoys some devil's staircase-like properties:

(i) Δ is strictly increasing,

(ii) at every irrational $\alpha > 0$, Δ is continuous and $\Delta(\alpha)$ is a transcendental number,

(iii) at every rational $\alpha > 0$, Δ is left-continuous but not right-continuous and $\Delta(\alpha)$ is an algebraic integer.

For an irrational α, one sees that $\Delta(\alpha)$ lies in $C_3 \cap Q_0^{(1)}$. An algebraic study of $\Delta(\alpha)$ for a rational α will appear in the subsequent paper [17].

We will also differentiate Δ wherever possible. Then apart from the interest in its own right the differentiation of Δ distinguishes, in an unexpected manner, some number theoretic properties of real numbers.

Let t be a real number and f be a positive increasing function defined on positive integers. We consider a set

$$D_f = \left\{ \frac{p}{q} \in \mathbb{Q} : 0 < \left| t - \frac{p}{q} \right| < \frac{1}{f(q)} \right\}.$$

Over some fixed class of functions, the supremum of functions f that make D_f infinite is called the *irrationality measure* of t. One of classic and challenging questions in number theory is which function f allows D_f to be finite or infinite. There is an extensive literature seeking to find the irrationality measures of some mathematical constants, e.g., [12, 22, 10, 14, 15, 20, 7]. Recently, Adamczewski and Cassaigne [4] obtained an effective upper bound of the irrationality measure of a real number whose integer base representation is generated by a finite automaton.

The class of functions we usually consider is that of monomials or that of exponential functions in q. The class of monomials defines the *irrationality exponent*, and the class of exponential functions defines the *irrationality base*. For instance, if t is a Liouville number then D_f is an infinite set for any monomial f and the irrationality exponent of t is infinite. In Section 5 we will also give examples of Liouville numbers whose irrationality base is any real number greater than 1. By definition, the irrationality measure of t measures how well t is approximable by rationals. In this sense Liouville numbers are well approximable by rationals.

Let us divide, according to the differentiation of Δ, the set of positive irrational numbers into three sets by

$$\mathbb{R}^{\geq 0} \backslash \mathbb{Q} = \{x : \Delta'(x) = 0\} \cup \{x : 0 < \Delta'(x) < \infty\} \cup \{x : \Delta \text{ is not differentiable at } x\}.$$

We will see that this trisection is related to the irrationality measure of x. Roughly speaking, if $x \in \mathbb{R}^{\geq 0} \setminus \mathbb{Q}$ is 'extremely' well approximable by rationals then Δ is not differentiable at x. Otherwise, if x is not 'too' well approximable by rationals then $\Delta'(x)$ exists and in most of such cases it is equal to zero. Since almost every real number is not well approximable by rationals [16], we can say that Δ is flat almost everywhere or its almost all increases are possible by discontinuous jumps only.

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It is worthwhile to mention here that Bullett and Sentenac [8] found a devil's staircase in a similar setting. The β-expansions that are mechanical words were also considered. But they fixed the base $\beta = 2$, and instead investigated the β-expansion $d_\beta(x)$ via varying x. More precisely, to given $\alpha \in (0,1)$, they assigned the T_β-orbit whose *rotation number* is equal to α. In fact they considered, though the way of construction, its inverse function to follow the usual definition of a devil's staircase: 'the graph of a non-constant function which is continuous, monotonic and locally constant on a set of full measure.' See [8] for details. The function Δ established in this paper is, however, nowhere constant and thus does not fit the definition of a devil's staircase. Its inverse function is indeed a devil's staircase in the sense of [8]. But we will not take its inverse. This makes it convenient to relate the differentiations of Δ with irrationality measures.

2 Christoffel words and a devil's staircase.

We recall the notations and concepts on language theory. Lothaire's book [18] will be a complete alternative to this brief review. Given a finite alphabet A, let A^* (resp. $A^{\mathbb{N}}$) be the set of finite (resp. infinite) words over A. A word $w \in A^* \cup A^{\mathbb{N}}$ is said to be a *factor* (resp. *prefix*, *suffix*) of a word $u \in A^* \cup A^{\mathbb{N}}$ provided u is expressed as $u = xwy$ (resp. $u = wy$, $u = xw$) for some words x and y. Note that if $u \in A^{\mathbb{N}}$ then so is y. In case $A \subset \mathbb{N}$, the usual *lexicographic order* on $A^{\mathbb{N}}$ has a natural extension to an order on $A^* \cup A^{\mathbb{N}}$ by substituting any $x \in A^*$ with $x0^\omega := x00\cdots$, even if $0 \notin A$. For instance, if $x, y \in A^*$ and $z \in A^{\mathbb{N}}$, then $x < y$ (resp. $y < z$, $z < y$) if and only if $x0^\omega < y0^\omega$ (resp. $y0^\omega < z$, $z < y0^\omega$). This total order is also called lexicographic order. A nonempty word $u \in A^*$ is *primitive* if $u = x^n$ for a nonempty x implies $n = 1$. For a word $u \in A^* \cup A^{\mathbb{N}}$, we mean by $\mathrm{alph}(u) \subset A$ the set of letters appearing in u. The set $A^{\mathbb{N}}$ is well endowed with a metric in a sense that the metric generates the usual product topology of $A^{\mathbb{N}}$ while A has the discrete topology. For any $x, y \in A^{\mathbb{N}}$, we define the distance between x and y by $d(x,y) = 2^{-n}$, where $n = \min\{k \geq 0 | x_k \neq y_k\}$.

For the present we suppose $A = \{0,1\}$ and consider mechanical words with their slope rational $\alpha = p/q \in [0,1]$, $\gcd(p,q) = 1$. It readily follows that $s_{\alpha,0}$ and $s'_{\alpha,0}$ are purely periodic. To be more precise, we define finite words $t_{p,q}$ and $t'_{p,q}$ by

$$t_{p,q} = a_0 \cdots a_{q-1}, \quad t'_{p,q} = a'_0 \cdots a'_{q-1},$$

where

$$a_i = \left\lfloor (i+1)\frac{p}{q} \right\rfloor - \left\lfloor i\frac{p}{q} \right\rfloor, \quad a'_i = \left\lceil (i+1)\frac{p}{q} \right\rceil - \left\lceil i\frac{p}{q} \right\rceil.$$

Then one has $s_{\alpha,0} = t_{p,q}^\omega$ and $s'_{\alpha,0} = t'_{p,q}{}^\omega$. These words $t_{p,q}$, $t'_{p,q}$ are said to be *Christoffel words*. We observe that $t_{1,1} = t'_{1,1} = 1$, and that if $p/q \neq 1$ then they are factored as

$$t_{p,q} = 0z_{p,q}1, \quad t'_{p,q} = 1z_{p,q}0,$$

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for some word $z_{p,q}$, called a *central word*. It is easy to see that all Christoffel words are primitive and that $z_{p,q}$ is a *palindrome word*, i.e., $z_{p,q}$ is equal to its reversal. We recall that if α is irrational, then

$$s_{\alpha,0} = 0c_\alpha \text{ and } s'_{\alpha,0} = 1c_\alpha$$

for some infinite word c_α, called the *characteristic word* of slope α. Before introducing our devil's staircase, we investigate lexicographic order between some variants of Christoffel words, which will play a crucial role in many contexts.

Lemma 2.1. *For $0 < \rho, \rho' \leq 1$, we have $s'_{\alpha,\rho} \leq s'_{\alpha,\rho'}$ if $\rho < \rho'$.*

Proof. See [18]. $\square$

Note here that if the value α is restricted to irrationals then we can say $s'_{\alpha,\rho} < s'_{\alpha,\rho'}$ if and only if $\rho < \rho'$.

We denote by σ the *shift* of finite or infinite sequences, and by $\{t\}$ the fractional part of t, i.e., $t = \lfloor t \rfloor + \{t\}$.

Proposition 2.2. *For integers $p, q > 0$, let $p/q \in (0,1)$ and $\gcd(p,q) = 1$. Then $1z_{p,q}1$ is lexicographically greater than all its proper suffixes.*

Proof. Note $s'_{p/q,0} = (1z_{p,q}0)^\omega$ and put $\alpha = p/q$. We define δ_0 by

$$\delta_0 = \min\{1 - \{\alpha n\} : 1 \leq n < q\}.$$

One finds that $0 < \delta_0 < 1$ if p and q are relatively prime. For $1 \leq n < q$, let $\sigma^n(1z_{p,q}1)$ be a proper suffix of $1z_{p,q}1$. Then this is also the prefix of $s'_{\alpha,\{\alpha n\}+\delta_0/2}$. Since $\{\alpha n\} + \delta_0/2 < 1$, we conclude from the lemma that

$$\sigma^n(1z_{p,q}1) < s'_{\alpha,\{\alpha n\}+\delta_0/2} \leq s'_{\alpha,0} = s'_{\alpha,1} < 1z_{p,q}1. \tag{1}$$

$\square$

Noting that any proper suffix of $1(z_{p,q}10)^\omega = 1z_{p,q}1(0z_{p,q}1)^\omega$ can be expressed as

$$\sigma^n(1(z_{p,q}10)^\omega) = s'_{p/q,\{pn/q\}+\delta_0/2}, \ n \geq 1,$$

we know that Inequality (1) shows the following.

Corollary 2.2.1. *$1(z_{p,q}10)^\omega$ is lexicographically greater than all its proper suffixes.*

In a β-shift S_β, Parry [19] noticed that $d_\beta(1)$ is quite distinguished in that $d_\beta(1)$ is greater than all its proper suffixes. Moreover he also showed that this property exhaustively determines whether a given sequence is $d_\beta(1)$ for some $\beta > 1$. Together with Parry's result, Proposition 2.2 guarantees the existence of a unique $\beta > 1$ such that $d_\beta(1) = 1z_{p,q}1$. This fact and [9] enable us to define the next function.

Theorem 2.3. *For integers a, b with $0 \le a < b$, let $f_{a,b}(1) = b + 1$. We have a function $f_{a,b} : (0,1] \to [b, b+1]$ which satisfies the following.*

(a) *If α is irrational, then $d_{f_{a,b}(\alpha)}(1) = bc_\alpha$. In other words, $f_{a,b}(\alpha)$-expansion of 1 is given by bc_α.*

(b) *If α is equal to p/q and p, q are relatively prime, then $d_{f_{a,b}(\alpha)}(1) = bz_{p,q}b$.*

Remark 2.4. Note that the characteristic words and the central words involved in Theorem 2.3 are the same as when $A = \{0, 1\}$ only except that 0 is replaced by a and 1 by b, that is, $\mathrm{alph}(c_\alpha) = \mathrm{alph}(z_{p,q}) = \{a, b\}$. Likewise, we will read below $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ as the corresponding words over an alphabet $A = \{a, b\}$.

From now on we suppose $a = b - 1$, so the alphabet is $A = \{a, b\} = \{b - 1, b\}$. We specify a function $\Delta : [0, \infty) \to \mathbb{R}$ in such a way that for any positive integer b, the restriction of Δ to the set $(b - 1, b]$ is defined to be the translation of $f_{b-1,b}$ by $b - 1$, that is,

$$\Delta|_{(b-1,b]}(x) := f_{b-1,b}(x - b + 1).$$

In addition we put $\Delta(0) := 1$. If we allowed the integer a to be less than $b - 1$, then three different letters might appear in $d_{f_{a,b}(\alpha)}(1)$, and hence many results to be obtained below would fail.

Lemma 2.5. *Suppose that a sequence $(a_n)_{n \ge 1}$ of real numbers is given to satisfy $a_1 \ge 1$, $a_1 a_2 \cdots \ne 10^\omega$ and $0 \le a_n \le M$, $n = 1, 2, \ldots$ for some M. Let ζ be the unique positive root of a series equation*

$$1 = \sum_{n=1}^{\infty} a_n x^n.$$

We also assume that ζ_m is the unique positive root of

$$1 = \sum_{n=1}^{\infty} b_{m,n} x^n,$$

where $0 \le b_{m,n} \le M$, $m, n = 1, 2, \ldots$, and where $b_{m,i} = a_i$, $1 \le i \le m$. Then we have

$$\lim_{m \to \infty} \zeta_m = \zeta.$$

Proof. By equating two series, we get

$$a_1(\zeta_m - \zeta) + a_2(\zeta_m^2 - \zeta^2) + \cdots + a_m(\zeta_m^m - \zeta^m) = \sum_{n=m+1}^{\infty} a_n \zeta^n - \sum_{n=m+1}^{\infty} b_{m,n} \zeta_m^n.$$

So one finds that

$$|\zeta_m - \zeta| \, |a_1 + a_2(\zeta_m + \zeta) + \cdots + a_m(\zeta_m^{m-1} + \cdots + \zeta^{m-1})| \le M \left(\sum_{n=m+1}^{\infty} \zeta^n + \sum_{n=m+1}^{\infty} \zeta_m^n \right),$$

which is followed by

$$0 \leq |\zeta_m - \zeta| \leq M \left(\frac{\zeta^{m+1}}{1 - \zeta} + \frac{\zeta_m^{m+1}}{1 - \zeta_m} \right). \tag{2}$$

The right-hand side of the inequality approaches zero as m tends to infinity because $0 < \zeta_m < 1 - \varepsilon$ for some $\varepsilon > 0$. $\qquad\square$

The next proposition determines the continuity of Δ.

Proposition 2.6. *For integers a, b with $0 \leq a = b - 1$, the function $f_{a,b}$ fulfills the following.*

(a) $f_{a,b}$ *is continuous at every irrational.*

(b) $f_{a,b}$ *is left-continuous but not right-continuous at every rational.*

Proof. First we suppose that α_1 is irrational, and define δ_N by

$$\delta_N = \min \left\{ \frac{\{\alpha_1 n\}}{n}, \frac{1 - \{\alpha_1 n\}}{n} : 1 \leq n \leq N \right\}. \tag{3}$$

Since α_1 is irrational, δ_N never vanishes for any $N \geq 1$. If $|\alpha - \alpha_1| < \delta_N$, then $s'_{\alpha,0}$ and $s'_{\alpha_1,0}$ have a common prefix of length N. Hence Lemma 2.5 shows that $|f_{a,b}(\alpha) - f_{a,b}(\alpha_1)|$ can be arbitrarily small if N is sufficiently large.

For nonzero $\alpha_0 = p/q \neq 1$, $\gcd(p, q) = 1$, we define δ_N by

$$\delta_N = \min \left\{ \frac{\{\alpha_0 n\}}{n} : 1 \leq n \leq N, \ n \not\equiv 0 \mod q \right\}. \tag{4}$$

Since $\gcd(p, q) = 1$, one has $\delta_N > 0$ for all $N \geq 1$. If $0 \leq \alpha_0 - \alpha < \delta_N$, then $s'_{\alpha,0}$ has a common prefix of length N with $s'_{\alpha_0,0} = (bz_{p,q}a)^\omega$. Given $\varepsilon > 0$, we can choose N so that $0 \leq \alpha_0 - \alpha < \delta_N$ implies

$$0 \leq f_{a,b}(\alpha_0) - f_{a,b}(\alpha) < \varepsilon.$$

Here we use the fact that if $\beta_0 = f_{a,b}(\alpha_0)$ and $d_{\beta_0}(1) = \epsilon_0 \epsilon_1 \cdots \epsilon_{q-1}$ then an equation $1 = \sum_{n=0}^\infty \epsilon'_n / \beta_0^{n+1}$ also holds, where $\epsilon'_n = \epsilon_n - 1$ for $n \equiv -1 \mod q$ and $\epsilon'_n = \epsilon_n$ otherwise.

To show that $f_{a,b}$ is not right-continuous, we assume $\alpha > \alpha_0 = p/q \neq 1$. Then one finds

$$s'_{\alpha,0} > b(z_{p,q}ba)^\omega > bz_{p,q}b > s'_{\alpha_0,0}.$$

With the aid of Corollary 2.2.1, we see that there exists $\gamma > 1$ such that

$$d_\gamma(1) = b(z_{p,q}ba)^\omega.$$

Then we have $f_{a,b}(\alpha) > \gamma > f_{a,b}(\alpha_0)$.

It remains to prove left continuity at 1. At first, one notes that $f_{a,b}(1) = b + 1$ is a positive root of $1 = \sum_{n=1}^\infty bx^{-n}$. So $0 \leq 1 - \alpha < 1/N$ implies that b^N is a prefix of $s'_{\alpha,0}$. Therefore, $b + 1 - f_{a,b}(\alpha)$ tends to zero as N increases. $\qquad\square$

The next theorem is easy consequences of the results above and [9].

Theorem 2.7. *We have the following.*

(a) Δ *is strictly increasing.*

(b) $\Delta(\mathbb{R}^+)$ *has Lebesgue measure zero.*

(c) *If $\alpha > 0$ is irrational, then*

$$d_{\Delta(\alpha)}(1) = bc_\alpha,$$

where $b = \lceil \alpha \rceil$ and $\mathrm{alph}(c_\alpha) = \{b - 1, b\}$. In other words, $\Delta(\alpha)$-expansion of 1 is given by bc_α. Moreover $\Delta(\alpha)$ is transcendental.

(d) *If the fractional part $\{\alpha\}$ of α is equal to p/q and p, q are relatively prime, then*

$$d_{\Delta(\alpha)}(1) = bz_{p,q}b,$$

where $b = \lceil \alpha \rceil$ and $\mathrm{alph}(z_{p,q}) = \{b - 1, b\}$. In this case, $\Delta(\alpha)$ is an algebraic integer.

(e) Δ *is continuous at every irrational point.*

(f) *At every rational point, Δ is left-continuous but not right-continuous.*

3   Self-Christoffel numbers.

In [9], $\beta > 1$ is called a *self-Sturmian number* if $d_\beta(1)$ is a Sturmian word. Now we consider its counterpart for Christoffel words.

Definition 3.1. For any rational $\alpha > 0$, the value $\Delta(\alpha)$ is called a *lower self-Christoffel number*, and the right limit $\Delta(\alpha+) := \lim_{x \to \alpha+} \Delta(x)$ called an *upper self-Christoffel number*.

By the definition lower self-Christoffel numbers are all (simple) beta-numbers and thus algebraic integers. This is also the case with upper self-Christoffel numbers.

Proposition 3.2. *Suppose $\alpha = b - 1 + p/q$, $0 < p \leq q$, $\beta_+ = \Delta(\alpha+)$ and $\gcd(p, q) = 1$. Then*

$$d_{\beta_+}(1) = b(z_{p,q}b(b - 1))^\omega$$

if $p/q \neq 1$, and

$$d_{\beta_+}(1) = (b + 1)b^\omega$$

if $p = q = 1$.

Proof. If $p/q \neq 1$, then we define δ_N by

$$\delta_N = \min\left\{\frac{1 - \{pn/q\}}{n} : 1 \leq n \leq N,\ n \not\equiv 0 \mod q\right\}. \tag{5}$$

Since p and q are relatively prime, one notes that $\delta_N > 0$ for all $N \geq 1$. Now an inequality $0 < \eta - \alpha < \delta_N$ implies that two words $s'_{\eta,0}$ and $b(z_{p,q}b(b-1))^\omega$ have a common prefix of length N. Now hence Corollary 2.2.1 and Lemma 2.5 prove $d_{\beta_+}(1) = b(z_{p,q}b(b-1))^\omega$.

If $p = q = 1$, then $0 < \eta - b < 1/N$ implies that $\mathrm{alph}(s'_{\eta,0}) = \{b+1, b\}$ and that $s'_{\eta,0}$ and $(b+1)b^\omega$ have the identical prefix of length N. The same reasoning as above also works. $\square$

We now introduce canonical integer polynomials considered by Parry [19], which have lower or upper self-Christoffel numbers as zeros. Although the proof is immediate we state this fact as a proposition. For a word $w = a_0 a_1 \cdots a_{n-1}$ with $a_i \in \mathbb{Z}$, we mean by $\vec{w}$, a vector $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n$.

Proposition 3.3. *Suppose that $\alpha = b - 1 + p/q$, $0 \leq p \leq q$, and $\gcd(p, q) = 1$ if $p \neq 0$. If $\beta = \Delta(\alpha)$ and $\beta_+ = \Delta(\alpha+)$, then β is a root of an equation*

$$x^q - \overrightarrow{bz_{p,q}b} \cdot (x^{q-1}, x^{q-2}, \ldots, 1) = 0,$$

and β_+ is a root of an equation

$$x^{q+1} - \overrightarrow{bz_{p,q}b} \cdot (x^q, x^{q-1}, \ldots, x) - x + 1 = 0.$$

Remark 3.4. Even if $q = 1$, we will, for convenience, abuse $bz_{p,q}b$ for b if $p = 0$ and for $b + 1$ if $p = 1$. Bearing this convention in mind, one can see that the preceding proposition covers the case where α is an integer.

4 Analysis on devil's staircase and irrationality measures

Two polynomials in the previous section make it possible to get some limit values of the devil's staircase Δ.

Lemma 4.1 ([19]). *Let β_n be the positive root of $1 = x^{-1} + x^{-n}$. Then*

(a) $\lim_{n \to \infty} \beta_n = 1$,

(b) $\lim_{n \to \infty} \frac{n}{\beta_n^n} = \infty$.

The above result is a key lemma for the next two propositions.

Proposition 4.2. *The following hold.*

(a) $\lim_{\alpha \to 0+} \Delta(\alpha) = 1$.
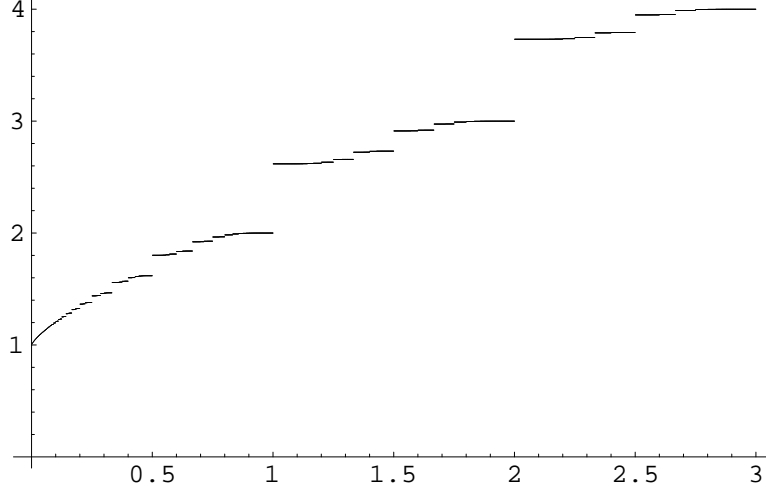
Figure 1: Devil's staircase

(b) $\Delta(b+) = \frac{b+2+\sqrt{b^2+4b}}{2}$ *for any integer* $b \geq 1$.

(c) $\lim_{\alpha \to \infty}(\Delta(\alpha) - \lceil \alpha \rceil) = 1$.

Proof. Part (a) and (b) are immediate. To prove Part (c), we put

$$\beta_q(b) := \Delta(b - 1 + q^{-1}).$$

If $b = \lceil \alpha \rceil$ and $b - 1 + q^{-1} < \alpha$, then

$$1 \geq \Delta(\alpha) - \lceil \alpha \rceil \geq \beta_q(b) - b.$$

Since $\beta_q(b)$ is a solution of $x^q - bx^{q-1} = (b-1)(x^{q-2} + \cdots + 1) + 1$, one has

$$\beta_q(b) - b = (b-1)\left(\frac{1}{\beta_q(b)} + \cdots + \frac{1}{\beta_q(b)^{q-1}}\right) + \frac{1}{\beta_q(b)^{q-1}}$$

$$= (b-1)\frac{1 - \frac{1}{\beta_q(b)^{q-1}}}{\beta_q(b) - 1} + \frac{1}{\beta_q(b)^{q-1}}.$$

As $\lim_{b \to \infty} \beta_q(b) = \infty$ and $b = \lfloor \beta_q(b) \rfloor$, we obtain the claim. $\square$

Although the graph of Δ is very ugly (see Figure 1), we can do some calculus explicitly. This function has derivatives at almost all irrational points, and left derivatives at all rational points. Basically, Δ cannot have right derivative at rational points since it is there discontinuous from the right, but we can say something like right derivative as the following proposition says. The proof employs a well-known fact, which is stated as a lemma for future references.

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Lemma 4.3. $\lim_{x\to a} f(x) = l$ if and only if $\lim_{n\to\infty} f(a_n) = l$ for any sequence $(a_n)_{n\geq 1}$ satisfying $a_n \neq a$ and $\lim_{n\to\infty} a_n = a$.

Proposition 4.4. The following hold for any rational $\alpha_0 > 0$.

(a) $\lim_{\alpha\to 0+} \frac{\Delta(\alpha)-1}{\alpha} = \infty$.

(b) $\lim_{\alpha\to\alpha_0-} \frac{\Delta(\alpha_0)-\Delta(\alpha)}{\alpha_0-\alpha} = 0$.

(c) $\lim_{\alpha\to\alpha_0+} \frac{\Delta(\alpha)-\Delta(\alpha_0+)}{\alpha-\alpha_0} = 0$.

Proof. (a) Let q be a positive integer and $\beta_{2q} := \Delta((2^{-1}q^{-1})+)$ be the upper self-Christoffel number. For any $\alpha \in (2^{-1}q^{-1}, q^{-1}]$, one finds

$$\frac{\Delta(\alpha)-1}{\alpha} > \frac{\beta_{2q}-1}{q^{-1}}.$$

Since β_{2q} is the positive root of $x^{2q+1} - x^{2q} - 2x + 1 = 0$, we have

$$\beta_{2q} - 1 = \frac{2\beta_{2q}-1}{\beta_{2q}^{2q}}.$$

Lemma 4.1 now shows that

$$\lim_{q\to\infty} q(\beta_{2q}-1) = \lim_{q\to\infty} \frac{q(2\beta_{2q}-1)}{\beta_{2q}^{2q}} = \lim_{q\to\infty} \frac{q}{\beta_{2q}^{2q}} = \infty.$$

If a positive sequence $(\alpha_n)_{n\geq 1}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, then

$$\lim_{n\to\infty} \frac{\Delta(\alpha_n)-1}{\alpha_n} = \infty.$$

Since $\bigcup_{q=1}^{\infty} (2^{-1}q^{-1}, q^{-1}] = (0,1]$, every α_n lies in some interval $(2^{-1}q^{-1}, q^{-1}]$.

Assume $\alpha_0 = p/q \neq 1$, $\gcd(p,q) = 1$ and define δ_N as Equation (4) for Part (b), and as Equation (5) for Part (c). In both cases, one observes that

$$\delta_N \geq \frac{1}{qN} \text{ and } \delta_{N+1} \geq \frac{1}{q(N+1)} \geq \frac{1}{2qN},$$

from which it follows that the infinite union $\bigcup_{N=1}^{\infty} ((2qN)^{-1}, \delta_N]$ is an interval $(0, \{\alpha_0\}]$ or $(0, 1-\{\alpha_0\}]$ in each case.

(b) If $(2qN)^{-1} < \alpha_0 - \alpha < \delta_N$, then $s'_{\alpha,0}$ and $s'_{\alpha_0,0}$ have a common prefix of length N. So Inequality (2) reads as follows:

$$\Delta(\alpha_0) - \Delta(\alpha) \leq (b+1)^3 \left(\frac{1}{(\Delta(\alpha)-1)\Delta(\alpha)^N} + \frac{1}{(\Delta(\alpha_0)-1)\Delta(\alpha_0)^N} \right)$$
$$< \frac{2(b+1)^3}{(c-1)c^N},$$

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for some $c > 1$ independent of N. Thus one obtains

$$\frac{\Delta(\alpha_0) - \Delta(\alpha)}{\alpha_0 - \alpha} < \frac{4qN(b+1)^3}{(c-1)c^N} \to 0, \quad \text{as } N \to \infty.$$

Given a sequence $(\alpha_n)_{n \geq 1}$ satisfying $\alpha_n < \alpha_0$ for $n \geq 1$ and $\lim_{n \to \infty} \alpha_n = \alpha_0$, we have now

$$\lim_{n \to \infty} \frac{\Delta(\alpha_0) - \Delta(\alpha_n)}{\alpha_0 - \alpha_n} = 0.$$

(c) If $(2qN)^{-1} < \alpha - \alpha_0 < \delta_N$, then $s'_{\alpha,0}$ and $b(z_{p,q}b(b-1))^\omega$ have a common prefix of length N. Now Inequality (2) implies that

$$\frac{\Delta(\alpha) - \Delta(\alpha_0+)}{\alpha - \alpha_0} \leq 2qN(b+1)^3 \left(\frac{1}{(\Delta(\alpha)-1)\Delta(\alpha)^N} + \frac{1}{(\Delta(\alpha_0+)-1)\Delta(\alpha_0+)^N} \right)$$

$$< \frac{4qN(b+1)^3}{(\Delta(\alpha_0+)-1)\Delta(\alpha_0+)^N} \to 0, \quad \text{as } N \to \infty.$$

If $\alpha_n > \alpha_0$ for $n \geq 1$ and $\lim_{n \to \infty} \alpha_n = \alpha_0$, then

$$\lim_{n \to \infty} \frac{\Delta(\alpha_n) - \Delta(\alpha_0)}{\alpha_n - \alpha_0} = 0.$$

We leave to the reader the proof when α_0 is an integer. $\qquad \square$

To find derivatives at irrational points is more involved, but uses essentially the same technique as in finding one side derivatives at rational points. We need some theory on Diophantine approximation.

We denote the regular continued fraction expansion of a real number t by

$$t = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

where the *partial quotients* a_i for $i \geq 1$ are positive integers. Then the i'th truncation $p_i/q_i := [a_0; a_1, \ldots, a_i]$ is called the i'th *convergent* of t.

Let t be a real number and p/q be a fraction in lowest terms with $q > 0$. Then we say that p/q is:

- a *best approximation of the first kind* to the number t if

$$\left| t - \frac{p}{q} \right| < \left| t - \frac{a}{b} \right|$$

  holds for any fraction $a/b \neq p/q$ with $0 < b \leq q$.

- a *best approximation of the second kind* to the number t if

$$|qt - p| < |bt - a|$$

  holds for any fraction $a/b \neq p/q$ with $0 < b \leq q$.

Because of an inequality

$$\left| t - \frac{p}{q} \right| < \frac{b}{q} \left| t - \frac{a}{b} \right| \le \left| t - \frac{a}{b} \right|,$$

every best approximation of the second kind is also a best approximation of the first kind. But the converse is not true in general. Regular continued fractions and Diophantine approximations have an intrinsic connection as the following two propositions say. See [16] for their proofs.

Proposition 4.5. *Suppose that t is a real number. Then*

(a) *every best approximation of the second kind to t is a convergent of t,*

(b) *if $\{t\} \ne 1/2$, then every convergent of t is a best approximation of the second kind to t.*

The next proposition states that if a/b is not a convergent of t then it can not be too close to t.

Proposition 4.6. *If p/q is a fraction in lowest terms that fulfills the inequality*

$$\left| t - \frac{p}{q} \right| < \frac{1}{2q^2},$$

then p/q is a convergent of t.

For a real number α, let $\|\alpha\|$ be the smallest distance from α to integers, namely,

$$\|\alpha\| := \min_{n \in \mathbb{Z}} |\alpha - n| = \min\{\{\alpha\}, 1 - \{\alpha\}\}.$$

If an irrational algebraic α is of degree n, then there exists a constant $c(\alpha) > 0$ depending only upon α such that $\|q\alpha\| > c(\alpha)q^{-n+1}$ for any nonzero $q \in \mathbb{Z}$. This fact is known as Liouville's Theorem. Transcendental numbers passing through this sieve are called *Liouville numbers*, e.g., $\sum_{k=1}^{\infty} 1/10^{k!}$. An irrational number α is said to have *irrationality exponent* $\mu(\alpha)$ if it is given by

$$\mu(\alpha) := \sup\{\nu : \liminf_{q \to \infty} q^{\nu-1}\|q\alpha\| = 0\}.$$

So we can say that Liouville numbers have their irrationality exponent infinity and vice versa. It is well known that every irrational number has its irrationality exponent greater than or equal to 2 and this value is in fact 2 almost everywhere [16]. In particular, real numbers whose irrationality exponents are equal to 2 include all irrational algebraic numbers [22]. For our purpose we need to look at Liouville numbers in more detail. For an irrational number α, a number

$$\theta(\alpha) := \sup\{\lambda : \liminf_{q \to \infty} \frac{\lambda^q}{q}\|q\alpha\| = 0\}$$

is called the *irrationality base* of α, which was coined by Sondow [23]. One can note the following facts from the definitions.

- $\mu(\alpha) < \infty$ *implies* $\theta(\alpha) = 1$.

- $\theta(\alpha) > 1$ *implies* $\mu(\alpha) = \infty$.

But there exists a real number α such that both $\mu(\alpha) = \infty$ and $\theta(\alpha) = 1$ hold at once. The irrationality exponent and base can be reformulated as follows.

Proposition 4.7. *For an irrational α, suppose that p_n/q_n is the n'th convergent. Then*

$$\mu(\alpha) = 1 - \liminf_{n \to \infty} \frac{\log \|q_n \alpha\|}{\log q_n},$$

$$\log \theta(\alpha) = - \liminf_{n \to \infty} \frac{\log \|q_n \alpha\|}{q_n}.$$

Proof. Put $\mu = \mu(\alpha)$ and $\theta = \theta(\alpha)$. We know that μ is the greatest upper bound of the set of all ν for which $0 < \|q\alpha\| < q^{-\nu+1}$ or

$$\nu < 1 - \frac{\log \|q\alpha\|}{\log q}$$

holds for infinitely many q. Thus one deduces that

$$\mu = \limsup_{q \to \infty} \left(1 - \frac{\log \|q\alpha\|}{\log q} \right) = 1 - \liminf_{q \to \infty} \frac{\log \|q\alpha\|}{\log q}.$$

Now we claim that

$$\liminf_{q \to \infty} \frac{\log \|q\alpha\|}{\log q} = \liminf_{n \to \infty} \frac{\log \|q_n \alpha\|}{\log q_n}.$$

If $\tau = \liminf_{q \to \infty} \log \|q\alpha\| / \log q$, then given any $\epsilon > 0$,

$$\tau - \epsilon < \frac{\log \|q\alpha\|}{\log q}$$

holds for all sufficiently large q. In particular,

$$\tau - \epsilon < \frac{\log \|q_n \alpha\|}{\log q_n}$$

holds for all sufficiently large n. On the other hand, for infinitely many q and n, we have

$$\tau + \epsilon > \frac{\log \|q\alpha\|}{\log q} > \frac{\log \|q_n \alpha\|}{\log q_n},$$

where $q_{n-1} \leq q < q_n$. Hence we conclude $\tau = \liminf_{n \to \infty} \log \|q_n \alpha\| / \log q_n$.

As for the irrationality base, θ is the greatest upper bound of the set of all λ for which $0 < \|q\alpha\| < q/\lambda^q$ or

$$\log \lambda < \frac{\log q}{q} - \frac{\log \|q\alpha\|}{q}$$

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holds for infinitely many q. The same reasoning as above proves that

$$\log \theta = \limsup_{q \to \infty} \left(\frac{\log q}{q} - \frac{\log \|q\alpha\|}{q} \right) = -\liminf_{q \to \infty} \frac{\log \|q\alpha\|}{q} = -\liminf_{n \to \infty} \frac{\log \|q_n \alpha\|}{q_n}.$$

$\square$

Corollary 4.7.1.

$$\mu(\alpha) = 1 + \limsup_{n \to \infty} \frac{\log q_{n+1}}{\log q_n},$$

$$\log \theta(\alpha) = \limsup_{n \to \infty} \frac{\log q_{n+1}}{q_n}.$$

Proof. These two equations follow from an inequality

$$\frac{1}{2q_{n+1}} < \|q_n \alpha\| < \frac{1}{q_{n+1}}.$$

$\square$

Remark 4.8. Corollary 4.7.1 was also noted by Sondow [23], but Proposition 4.7 indeed verifies his proof.

Although Liouville numbers are already well approximable by rationals, we make the scale of well approximability more minutely. We now trisect Liouville numbers according to their approximability by rationals.

Definition 4.9. Let α be a Liouville number. We say that α is

  (a) *hypo-exponential* if $\mu(\alpha) = \infty$ and $\theta(\alpha) = 1$,

  (b) *exponential* if $1 < \theta(\alpha) < \infty$,

  (c) *hyper-exponential* if $\theta(\alpha) = \infty$.

In terms of T_β-orbit the value of Δ is, compared with the other real numbers, quite exceptional. This point was illuminated in [9] for irrationals, and in [17] for rationals.

Lemma 4.10 ([9, 17])**.** *Let $\alpha > 0$ be real and $\beta = \Delta(\alpha)$. Then for all integers $k \geq 1$,*

$$1 - \frac{1}{\beta} < T_\beta^k(1) < 1,$$

unless $T_\beta^k(1) = 0$.

If $d_\beta(1)$ is finite, say $d_\beta(1) = a_1 \cdots a_q$, then β is the unique positive root of an equation

$$1 = \sum_{n=1}^{q} \frac{a_n}{x^n}.$$

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But β is also the root of a series equation

$$1 = \sum_{n=1}^{\infty} \frac{a'_n}{x^n}, \quad a'_n = \begin{cases} a_n - 1 & \text{if } n \equiv 0 \mod q, \\ a_n & \text{otherwise.} \end{cases}$$

For convenience' sake, we introduce a notation.

$$d_\beta(1-) := \lim_{x \to 1-} d_\beta(x) = \begin{cases} d_\beta(1), & \text{if } d_\beta(1) \text{ is infinite,} \\ (a_1 \cdots a_{q-1}(a_q - 1))^\omega, & \text{if } d_\beta(1) = a_1 \cdots a_q. \end{cases}$$

The next lemma will show below that there exists some irrational number where Δ is not differentiable.

Lemma 4.11. *Given an irrational $\alpha > 0$ and a real $\alpha_N > 0$, let $\beta = \Delta(\alpha)$ and $\beta_N = \Delta(\alpha_N)$. If $d_\beta(1-)$ and $d_{\beta_N}(1-)$ have a common prefix of length $N-1$ and their N'th letters are different, then there exist constants $\delta \geq 0$ and $c > 0$ independent of N such that the following inequality holds:*

$$|\beta - \beta_N| > \frac{c}{N(\beta + \delta)^N}.$$

Proof. Let $d_\beta(1-) = a_1 a_2 \cdots$ and $d_{\beta_N}(1-) = b_1 b_2 \cdots$. Suppose at first $\alpha > \alpha_N$. We find then that

$$\frac{a_1}{\beta} + \cdots + \frac{a_{N-1}}{\beta^{N-1}} + \frac{a_N}{\beta^N} + \frac{a_{N+1}}{\beta^{N+1}} + \cdots = \frac{b_1}{\beta_N} + \cdots + \frac{b_{N-1}}{\beta_N^{N-1}} + \frac{b_N}{\beta_N^N} + \frac{b_{N+1}}{\beta_N^{N+1}} + \cdots.$$

Since $a_1 = b_1, \ldots, a_{N-1} = b_{N-1}$ and $a_N = b_N + 1$, one has

$$a_1 \left(\frac{1}{\beta_N} - \frac{1}{\beta} \right) + \cdots + a_{N-1} \left(\frac{1}{\beta_N^{N-1}} - \frac{1}{\beta^{N-1}} \right) + a_N \left(\frac{1}{\beta_N^N} - \frac{1}{\beta^N} \right) \quad (6)$$

$$> \frac{a_{N+1}}{\beta^{N+1}} + \frac{a_{N+2}}{\beta^{N+2}} + \cdots > \left(1 - \frac{1}{\beta} \right) \frac{1}{\beta^N}, \quad (7)$$

where the last inequality follows from Lemma 4.10. On the other hand, (6) is bounded above as

$$a_1 \left(\frac{1}{\beta_N} - \frac{1}{\beta} \right) + \cdots + a_N \left(\frac{1}{\beta_N^N} - \frac{1}{\beta^N} \right)$$

$$= \left(\frac{1}{\beta_N} - \frac{1}{\beta} \right) \left(a_1 + a_2 \left(\frac{1}{\beta_N} + \frac{1}{\beta} \right) + \cdots + a_N \left(\frac{1}{\beta_N^{N-1}} + \cdots + \frac{1}{\beta^{N-1}} \right) \right)$$

$$< bN \left(\frac{1}{\beta_N} - \frac{1}{\beta} \right) \sum_{n=0}^{\infty} \frac{1}{\beta_N^i} = \frac{bN(\beta - \beta_N)}{\beta(\beta_N - 1)},$$

where $b = \lceil \alpha \rceil$. Therefore we get

$$\beta - \beta_N > \frac{(\beta - 1)(\beta_N - 1)}{bN\beta^N}.$$

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If $\alpha < \alpha_N$, then we have $b_N = a_N + 1$. So one can show

$$\frac{bN(\beta_N - \beta)}{\beta_N(\beta - 1)} > b_1\left(\frac{1}{\beta} - \frac{1}{\beta_N}\right) + \cdots + b_N\left(\frac{1}{\beta^N} - \frac{1}{\beta_N^N}\right)$$
$$> \frac{b_{N+1}}{\beta_N^{N+1}} + \frac{b_{N+2}}{\beta_N^{N+2}} + \cdots > \left(1 - \frac{1}{\beta_N}\right)\frac{1}{\beta_N^N}.$$

Hence we have

$$\beta_N - \beta > \frac{(\beta - 1)(\beta_N - 1)}{bN\beta_N^N} > \frac{(\beta - 1)(\beta_N - 1)}{bN(\beta + \delta)^N},$$

for some $\delta > 0$. $\qquad\square$

The next theorem enables us to differentiate Δ almost everywhere. The proof depends on the completeness of real numbers.

Theorem 4.12. *Suppose that α_0 is an irrational number.*

(a) If $\theta(\alpha_0) < \Delta(\alpha_0)$ then $\Delta'(\alpha_0) = 0$.

(b) If $\theta(\alpha_0) > \Delta(\alpha_0)$ then $\Delta(x)$ is not differentiable at $x = \alpha_0$.

Proof. (a) For an integer $N \geq 1$, we define δ_N by Equation (3) or equivalently by

$$\delta_N = \min\left\{\frac{\|\alpha_0 n\|}{n} : 1 \leq n \leq N\right\}.$$

Let us fix a real $\lambda > 1$ with $\theta(\alpha_0) < \lambda < \Delta(\alpha_0)$. Then there exists an integer $N_0 \geq 1$ such that $\delta_n > \lambda^{1-n}$ holds for every $n \geq N_0$. Indeed, suppose otherwise that $L = \{n \in \mathbb{N} : \delta_n \leq \lambda^{1-n}\}$ is an infinite set. Then so is $\{m_n : \delta_n = \|\alpha_0 m_n\|/m_n,\ m_n \leq n,\ n \in L\}$ — otherwise, α_0 is a rational number. For a real κ with $\theta(\alpha_0) < \kappa < \lambda$ and $n \in L$, we have

$$\kappa^{m_n}\frac{\|\alpha_0 m_n\|}{m_n} = \kappa^{m_n}\delta_n \leq \frac{\kappa^{m_n}}{\lambda^{n-1}} = \frac{1}{\lambda^{n-m_n-1}}\frac{\kappa^{m_n}}{\lambda^{m_n}} \leq \lambda\frac{\kappa^{m_n}}{\lambda^{m_n}} \to 0, \quad \text{as } m_n \to \infty,$$

which means $\theta(\alpha_0) \geq \kappa$, a contradiction. Therefore if $n \geq N_0$ then two intervals $(\lambda^{-n}, \delta_n]$ and $(\lambda^{-(n+1)}, \delta_{n+1}]$ have a nonempty overlap, and so we note

$$\bigcup_{n=N_0}^{\infty} (\lambda^{-n}, \delta_n] = (0, \delta_{N_0}].$$

One sees that for all α satisfying $\lambda^{-n} < |\alpha - \alpha_0| < \delta_n$, the words $s'_{\alpha,0}$ and $s'_{\alpha_0,0}$ have a common prefix of length n. Here we may assume $\lambda < \Delta(\alpha)$ for sufficiently large n. Hence Lemma 2.5 guarantees some constant $\gamma > 1$ with $\lambda < \gamma < \Delta(\alpha_0)$, for which

$$|\Delta(\alpha) - \Delta(\alpha_0)| \leq (b+1)^3\left(\frac{1}{(\Delta(\alpha) - 1)\Delta(\alpha)^n} + \frac{1}{(\Delta(\alpha_0) - 1)\Delta(\alpha_0)^n}\right)$$
$$< \frac{2(b+1)^3}{(\gamma - 1)\gamma^n},$$

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where $b = \lceil \alpha_0 \rceil$. Note that γ is independent of n. One thus finds that

$$\frac{|\Delta(\alpha) - \Delta(\alpha_0)|}{|\alpha - \alpha_0|} < \frac{2(b+1)^3 \lambda^n}{(\gamma - 1)\gamma^n} \to 0, \quad \text{as } n \to \infty. \tag{8}$$

If $(\alpha_k)_{k \geq 1}$ is a sequence such that $\alpha_k \neq \alpha_0$ for $k \geq 1$ and $\lim_{k \to \infty} \alpha_k = \alpha_0$ then the value $|\alpha_k - \alpha_0|$ eventually lies in an interval out of $\{(\lambda^{-n}, \delta_n] : n \geq N_0\}$, whence we get

$$\lim_{k \to \infty} \frac{|\Delta(\alpha_k) - \Delta(\alpha_0)|}{|\alpha_k - \alpha_0|} = 0.$$

Now Lemma 4.3 proves the claim.

(b) Adopting δ_N as Part (a), we know that if $\Delta(\alpha_0) < \lambda < \theta(\alpha_0)$ then the inequality $\|\alpha_0 N\|/N < \lambda^{-N}$ and thus $\delta_N < \lambda^{-N}$ hold for infinitely many N. Let us fix λ in the interval $(\Delta(\alpha_0), \theta(\alpha_0))$ and collect such N's by

$$Q = \{n \in \mathbb{N} : \delta_n \leq \frac{\|\alpha_0 n\|}{n} < \lambda^{-n}\}.$$

For sufficiently large $q \in Q$, one can find an integer p for which the inequality

$$\frac{\|\alpha_0 q\|}{q} = \left| \alpha_0 - \frac{p}{q} \right| < \frac{1}{\lambda^q} < \frac{1}{2q^2}$$

holds. Hence Proposition 4.6 shows that if p, q are relatively prime then p/q is a convergent of α_0. Otherwise if $d = \gcd(p, q)$ and $p = dp', q = dq'$ then one has

$$\left| \alpha_0 - \frac{p'}{q'} \right| < \frac{1}{\lambda^{dq'}} < \frac{1}{\lambda^{q'}},$$

which implies $q' \in Q$. We claim that infinitely many $q \in Q$ are indeed the denominators of convergents of α_0. If this is not the case, then there exist an integer $q \in \mathbb{N}$ and an integer sequence $(d_n)_{n \geq 1}$ such that $\lim_{n \to \infty} d_n = \infty$ and $d_n q \in Q$ for all $n \geq 1$. In other words, we have for some integer p,

$$\left| \alpha_0 - \frac{p}{q} \right| < \frac{1}{\lambda^{d_n q}}, \ n = 1, 2, \ldots.$$

But this inequality forces α_0 to be equal to p/q. This is a contradiction.

Now suppose that $q_1 < q_2 < \cdots$ are the denominators of the convergents p_i/q_i of α_0 and that every q_i lies in Q. Then Proposition 4.5 leads us to deduce that the fraction p_i/q_i is a best approximation of the second kind, and hence of the first kind, i.e.,

$$\delta_{q_i} = \frac{\|\alpha_0 q_i\|}{q_i} \quad \text{and} \quad \delta_{q_i} < \delta_{q_i - 1}.$$

For each $i \geq 1$, we define α_{q_i} by

$$\alpha_{q_i} := \begin{cases} \alpha_0 - \delta_{q_i} & \text{if } \|q_i \alpha_0\| = q_i \alpha_0 - p_i, \\ \alpha_0 + \min\{\frac{\delta_{q_i} + \delta_{q_i - 1}}{2}, 2\delta_{q_i}\} & \text{if } \|q_i \alpha_0\| = p_i - q_i \alpha_0. \end{cases}$$

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Then one notes that

$$\lim_{i \to \infty} \alpha_{q_i} = \alpha_0 \ \ \text{and} \ \ |\alpha_{q_i} - \alpha_0| \le 2\delta_{q_i} < \frac{2}{\lambda^{q_i}},$$

since q_i is an element of Q. Let us assume $\beta_0 := \Delta(\alpha_0)$, $\beta_{q_i} := \Delta(\alpha_{q_i})$ and $d_{\beta_0}(1) = a_1 \cdots a_{q_i-1} a_{q_i} \cdots$. On the one hand if $\|q_i \alpha_0\| = q_i \alpha_0 - p_i$ then β_{q_i} is a lower self-Christoffel number, more precisely,

$$d_{\beta_{q_i}}(1-) = (a_1 \cdots a_{q_i-1}(a_{q_i} - 1))^\omega.$$

On the other hand if $\|q_i \alpha_0\| = p_i - q_i \alpha_0$, then a word $a_1 \cdots a_{q_i-1}(a_{q_i} + 1)$ is a prefix of $d_{\beta_{q_i}}(1-)$. In both cases, Lemma 4.11 shows that if i is sufficiently large then

$$|\Delta(\alpha_{q_i}) - \Delta(\alpha_0)| > \frac{c}{q_i \gamma^{q_i}},$$

for some constants $c > 0$ and $\Delta(\alpha_0) \le \gamma < \lambda$. Since γ is eventually independent of i, we conclude the following:

$$\frac{|\Delta(\alpha_{q_i}) - \Delta(\alpha_0)|}{|\alpha_{q_i} - \alpha_0|} > \frac{c\lambda^{q_i}}{2q_i \gamma^{q_i}} \to \infty, \quad \text{as } i \to \infty.$$

$\square$

In the case of $\theta(\alpha_0) > \Delta(\alpha_0)$, we can not say that $\Delta'(\alpha_0) = \infty$ as in Proposition 4.4 (a). The proof of Part (b) only shows the existence of a sequence $(\alpha_n)_{n \ge 1}$ converging to α_0 for which $\lim_{n \to \infty} |\Delta(\alpha_n) - \Delta(\alpha_0)|/|\alpha_n - \alpha_0| = \infty$ holds.

Theorem 4.12 covers two trivial cases where $\theta(\alpha_0) = 1$ and $\theta(\alpha_0) = \infty$.

Corollary 4.12.1. *Suppose that $\alpha_0 > 0$ is irrational.*

(a) *If α_0 is either a non-Liouville or a hypo-exponential Liouville number, then $\Delta'(\alpha_0) = 0$.*

(b) *If α_0 is a hyper-exponential Liouville number, then $\Delta(x)$ is not differentiable at $x = \alpha_0$.*

A real valued function $f(x)$ is said to satisfy the *Lipschitz condition* of *order* η at $x = x_0$ if η is the supremum of all ζ for which

$$|f(x) - f(x_0)| \le C|x - x_0|^\zeta, \ \ C : \text{constant}$$

holds on some open interval containing x_0. Since the function $\Delta(x)$ is non-differentiable on a dense subset of $[0, \infty)$, its second derivative is nonsense everywhere. But this 'local' Lipschitz condition allows us to measure how flat $\Delta(x)$ is wherever it is differentiable. The next theorem says that Δ is totally flat almost everywhere.

Theorem 4.13. *Assume that α_0 is an irrational number.*

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(a) *If α_0 is either a non-Liouville or a hypo-exponential Liouville number, then $\Delta(x)$ satisfies the Lipschitz condition of infinite order at $x = \alpha_0$.*

(b) *If $1 < \theta(\alpha_0) < \Delta(\alpha_0)$, then $\Delta(x)$ satisfies the Lipschitz condition of order $\log \Delta(\alpha_0)/\log \theta(\alpha_0)$ at $x = \alpha_0$.*

Proof. Suppose $\theta(\alpha_0) < \Delta(\alpha_0)$. Let λ and γ be chosen as in the proof of Theorem 4.12 (a). So we have $\theta(\alpha_0) < \lambda < \gamma < \Delta(\alpha_0)$. From Inequality (8) it follows that

$$\frac{|\Delta(\alpha) - \Delta(\alpha_0)|}{|\alpha - \alpha_0|^\zeta} < \frac{2(b+1)^3 \lambda^{\zeta n}}{(\gamma - 1)\gamma^n}.$$

Now $\lim_{n \to \infty} \lambda^{\zeta n}/\gamma^n = 0$ if and only if $\zeta < \log \gamma/\log \lambda$. Letting $\lambda \searrow \theta(\alpha_0)$ and $\gamma \nearrow \Delta(\alpha_0)$, we obtain the claim. $\qquad\square$

5 Examples

In this section we give some Liouville numbers that belong to each of three classes, which make us believe that the derivative of Δ assumes zero at every non-pathological constant. Some of these examples can be found in [23].

Example 1. $\Delta'(\sqrt{2}) = \Delta'(e) = \Delta'(\pi) = \Delta'(\log 2) = \Delta'(\zeta(3)) = \Delta'(\Gamma(1/4)) = \Delta'(\Gamma(1/3)) = \Delta'(2^{\sqrt{2}}) = \Delta'(\log 3/\log 2) = \Delta'(e^\pi) = 0$, where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ for } s > 1,$$

and

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \text{ for } s > 0.$$

Proof. Liouville Theorem tells us that all algebraic irrational numbers have finite irrationality exponents. Euler explicitly found the continued fraction of e by

$$e = [2; \overline{1, 2m, 1}]_{m \geq 1} := [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots].$$

With Corollary 4.7.1, one can show $\mu(e) < \infty$. Or more explicitly it is known [10] that if $\epsilon > 0$ and $p, q \in \mathbb{N}$ with $q > q_0(\epsilon)$ for some constant q_0 then

$$\left| e - \frac{p}{q} \right| > \left(\frac{1}{2} - \epsilon \right) \frac{\log \log q}{q^2 \log q}.$$

In [15, 14], Hata showed that π and $\log 2$ have finite irrationality exponents, explicitly,

$$\mu(\pi) \leq 8.0161, \ \mu(\log 2) \leq 3.8914.$$

As for $\zeta(3)$ the most up-to-date result due to Rhin and Viola [20] is

$$\mu(\zeta(3)) \leq 5.513891.$$

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In 2002, Bruiltet [7] obtained the explicit upper bounds of irrationality measures for $\Gamma(1/4)$ and $\Gamma(1/3)$:

$$\mu(\Gamma(1/4)) \leq 10^{143}, \ \mu(\Gamma(1/3)) \leq 10^{151}.$$

Whether e^π is a Liouville number is an open problem but the known estimate given in [25]

$$\left| e^\pi - \frac{p}{q} \right| > e^{-c(\log q)(\log\log q)}, \ c = 2^{60}, \ q \geq 3$$

is sufficient enough to show that the irrationality base for e^π is 1. Indeed since for any $\lambda > 1$

$$\frac{\lambda^q}{e^{c(\log q)(\log\log q)}} > \frac{\lambda^q}{e^{c(\log q)^2}}$$

tends to infinity as q increases, we see that $\theta(e^\pi) = 1$. Gelfond Theorem [12] implies that for any $\epsilon > 0$ there exist some constant $c_1, c_2 > 1$ such that

$$\left| 2^{\sqrt{2}} - \frac{p}{q} \right| > (c_1 q)^{-(\log\log(c_1 q))^{5+\epsilon}}$$

and

$$\left| \frac{\log 3}{\log 2} - \frac{p}{q} \right| > e^{-(\log(c_2 q))^{4+\epsilon}}.$$

Along the same line as e^π, we can show that $\theta(2^{\sqrt{2}}) = \theta(\log 3/\log 2) = 1$. $\quad\square$

A real number is said to be *automatic* if its integer base representation is generated by a finite automaton. For details the reader is referred to [4].

Example 2. Let $\alpha > 0$ be an automatic real number. Then $\Delta'(\alpha) = 0$.

This example follows from the next theorem.

Theorem 5.1 ([4])**.** *Assume that*

$$\{\alpha\} = \sum_{k=1}^{\infty} \frac{a_k}{b^k}$$

is an expansion of the fractional part of $\alpha > 0$ in base $b \in \mathbb{N}$, and that the sequence $(a_k)_{k \geq 1}$ is generated by a finite automaton. Then

$$\mu(\alpha) \leq C,$$

where the constant C depends on the automaton giving $(a_k)_{k \geq 1}$.

Example 3. $\sum_{k=1}^{\infty} 1/10^{k!}$ is a hypo-exponential Liouville number.

Proof. Let $\alpha_1 = \sum_{k=1}^{\infty} 1/10^{k!}$. It is well known that α_1 is a Liouville number. See for instance [13]. Let $(p_n/q_n)_{n \geq 1}$ be the sequence of the n'th convergents of α_1, and let

$$\frac{r_m}{s_m} := \sum_{k=1}^{m} \frac{1}{10^{k!}}$$

be fractions in lowest terms, i.e., $\gcd(r_m, s_m) = 1$. One can note then that $s_m = 10^{m!}$. From an inequality

$$\alpha_1 - \frac{r_m}{s_m} = \frac{1}{10^{(m+1)!}} + \frac{1}{10^{(m+2)!}} + \cdots$$

$$< \frac{1}{10^{(m+1)!}} \left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right) = \frac{10}{9 \cdot 10^{(m+1)!}} < \frac{1}{2s_m^2}, \quad m = 2, 3, 4, \ldots,$$

it follows that $(r_m/s_m)_{m \geq 2}$ is a subsequence of $(p_n/q_n)_{n \geq 1}$. We assume that a subsequence $(q_{n_i})_{i \geq 1}$ satisfies an equality

$$\lim_{i \to \infty} \frac{\log q_{n_i+1}}{q_{n_i}} = \log \theta(\alpha_1).$$

Then one can choose a (sub)sequence $\{s_{m_i}\}_{i \geq 1}$ of $\{s_m\}_{m \geq 2}$ so that $s_{m_i} \leq q_{n_i}$ and $q_{n_i+1} \leq s_{m_i+1}$ hold. Note here that we should allow some m_i to be equal to m_{i+1}. We have therefore

$$\log \theta(\alpha_1) = \lim_{i \to \infty} \frac{\log q_{n_i+1}}{q_{n_i}} \leq \lim_{i \to \infty} \frac{\log s_{m_i+1}}{s_{m_i}} = \lim_{i \to \infty} \frac{(m_i+1)! \log 10}{10^{m_i!}} = 0.$$

$\square$

For a nonnegative integer k, we denote by $\mathcal{EXP}_k(x)$ the k-fold power of x, for example,

$$\mathcal{EXP}_0(x) = 1, \ \mathcal{EXP}_1(x) = x, \ \mathcal{EXP}_2(x) = x^x, \ \mathcal{EXP}_3(x) = x^{x^x}, \ldots.$$

Let $\alpha = [a_0; a_1, a_2, \ldots]$ be irrational. From continued fraction theory, we know that the more rapidly a_n increases, the more approximable by rationals α is. Although the sequence $a_n = \mathcal{EXP}_n(10)$ increases extremely fast, it is not enough for $\alpha_2 = [a_0; a_1, a_2, \ldots]$ to be an exponential or hyper-exponential Liouville number. Since $q_{n+1} = a_{n+1}q_n + q_{n-1}$, we see that $a_{n+1}q_n < q_{n+1} < (a_{n+1}+1)q_n$ and hence

$$\prod_{i=1}^{n} a_i < q_n < \prod_{i=1}^{n} (a_i + 1).$$

Example 4. $[\mathcal{EXP}_0(10); \mathcal{EXP}_1(10), \mathcal{EXP}_2(10), \ldots]$ is a hypo-exponential Liouville number.

Proof. Suppose $a_n = \mathcal{EXP}_n(10)$ and $\alpha_2 = [a_0; a_1, a_2, \ldots]$. First α_2 is actually a Liouville number. This is because

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n} \geq \limsup_{n\to\infty} \frac{\log a_{n+1} + \log \prod_{i=1}^{n} a_i}{\log \prod_{i=1}^{n}(a_i+1)} \geq \limsup_{n\to\infty} \frac{\log a_{n+1}}{\log(2^n a_n^n)}$$

$$= \limsup_{n\to\infty} \frac{a_n \log 10}{n(\log 2 + \log a_n)} \geq \limsup_{n\to\infty} \frac{10^{a_{n-1}}}{2n a_{n-1} \log 10} = \infty.$$

To show that α_2 is hypo-exponential is also similar to the above. We get

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n} \leq \limsup_{n\to\infty} \frac{\log \prod_{i=1}^{n+1}(a_i+1)}{\prod_{i=1}^{n} a_i} \leq \limsup_{n\to\infty} \frac{\log(2^{n+1} a_{n+1}^{n+1})}{\prod_{i=1}^{n} a_i}$$

$$= \limsup_{n\to\infty} \frac{(n+1)(\log 2 + a_n \log 10)}{\prod_{i=1}^{n} a_i} = 0.$$

$\square$

From now on we construct exponential Liouville numbers.

Let a sequence $(a_n)_{n \geq 1}$ be defined by

$$a_1 = 10, \ a_{n+1} = a_n!, \ n \geq 1,$$

and we denote a_{n+1} by $10!^n$. Then a number $[0; 10, 10!, 10!^2, 10!^3, \ldots]$ still fails to be hyper-exponential.

Example 5. $\alpha_3 = [0; 10, 10!, 10!^2, 10!^3, \ldots]$ is an exponential Liouville number. More precisely,

$$2.9221 < \theta(\alpha_3) < 7.9433.$$

Proof. At first we note that

$$\log \theta(\alpha_3) = \limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n} = \limsup_{n\to\infty} \frac{\log(a_{n+1} q_n + q_{n-1})}{q_n}$$

$$= \limsup_{n\to\infty} \frac{\log a_{n+1} + \log q_n + \log\left(1 + \frac{q_{n-1}}{a_{n+1} q_n}\right)}{q_n}$$

$$= \limsup_{n\to\infty} \frac{\log a_{n+1}}{q_n} = \limsup_{n\to\infty} \frac{\log(a_n!)}{q_n},$$

and that

$$n \log n - n \leq \int_{1}^{n} \log x \ dx \leq \log n! \leq (n-1) \log n.$$

Hence we find

$$\log \theta(\alpha_3) \leq \limsup_{n\to\infty} \frac{(a_n - 1) \log a_n}{\prod_{i=1}^{n} a_i} \leq \limsup_{n\to\infty} \frac{(a_n - 1)(a_{n-1} - 1) \log a_{n-1}}{\prod_{i=1}^{n} a_i}$$

$$\leq \cdots \leq \limsup_{n\to\infty} \frac{(\prod_{i=1}^{n}(a_i - 1)) \log a_1}{\prod_{i=1}^{n} a_i} = \limsup_{n\to\infty} \log a_1 \prod_{i=1}^{n}\left(1 - \frac{1}{a_i}\right)$$

$$< \frac{9}{10} \log 10.$$

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To the other direction, one can derive

$$\log \theta(\alpha_3) \geq \limsup_{n \to \infty} \frac{a_n \log a_n - a_n}{\prod_{i=1}^{n}(a_i + 1)} \geq \limsup_{n \to \infty} \frac{a_n a_{n-1} \log a_{n-1} - a_n a_{n-1} - a_n}{\prod_{i=1}^{n}(a_i + 1)}$$

$$\geq \cdots \geq \limsup_{n \to \infty} \frac{(a_1 \cdots a_n) \log a_1 - [(a_1 \cdots a_n) + (a_2 \cdots a_n) + \cdots + a_n]}{\prod_{i=1}^{n}(a_i + 1)}$$

$$= \limsup_{n \to \infty} \frac{\log a_1 - [1 + a_1^{-1} + (a_1 a_2)^{-1} + \cdots + (a_1 \cdots a_{n-1})^{-1}]}{\prod_{i=1}^{n}(1 + a_i^{-1})}.$$

Now from

$$1 + a_1^{-1} + (a_1 a_2)^{-1} + \cdots + (a_1 \cdots a_{n-1})^{-1} < \sum_{i=0}^{\infty} \frac{1}{10^i} = \frac{10}{9}$$

and from

$$\prod_{i=1}^{n} \left(1 + \frac{1}{a_i}\right) \leq \prod_{i=1}^{n} \left(1 + \frac{1}{10^{2^{i-1}}}\right) = \sum_{i=0}^{\infty} \frac{1}{10^i} = \frac{10}{9},$$

it follows that

$$\log \theta(\alpha_3) > \frac{9}{10} \log 10 - 1.$$

Gathering both results we have

$$10^{0.9} e^{-1} < \theta(\alpha_3) < 10^{0.9}.$$

$\square$

Remark 5.2. The author does not know what value the irrationality base of α_3 has.

Next, we give an exponential Liouville number that is represented by series. Let

$$\alpha_4 = \sum_{k=1}^{\infty} \frac{1}{\mathcal{EXP}_k(10)} = \frac{1}{10} + \frac{1}{10^{10}} + \frac{1}{10^{10^{10}}} + \cdots.$$

Example 6. $\theta(\alpha_4) = 10$. So $\Delta'(9 + \alpha_4) = 0$ but $\Delta(x)$ is not differentiable at $x = 8 + \alpha_4$.

Proof. Let $(p_n/q_n)_{n \geq 1}$ be the sequence of the n'th convergents of α_4, and let

$$\frac{r_m}{s_m} := \sum_{k=1}^{m} \frac{1}{\mathcal{EXP}_k(10)}$$

be fractions in lowest terms. Then arguments similar to those of Example 2 show that $s_m = \mathcal{EXP}_m(10)$ and

$$\alpha_4 - \frac{r_m}{s_m} < \frac{10}{9 \cdot \mathcal{EXP}_{m+1}(10)},$$

which in turn implies that $(r_m/s_m)_{m\geq 1}$ is a subsequence of $(p_n/q_n)_{n\geq 1}$. If $(q_{n_i})_{i\geq 1}$ satisfies

$$\lim_{i\to\infty} \frac{\log q_{n_i+1}}{q_{n_i}} = \log\theta(\alpha_4),$$

then as in Example 3 there exists a (sub)sequence $(s_{m_i})_{i\geq 1}$ such that $s_{m_i} \leq q_{n_i}$ and $q_{n_i+1} \leq s_{m_i+1}$ hold. So one can derive

$$\log\theta(\alpha_4) = \lim_{i\to\infty} \frac{\log q_{n_i+1}}{q_{n_i}} \leq \lim_{i\to\infty} \frac{\log s_{m_i+1}}{s_{m_i}} = \lim_{i\to\infty} \frac{s_{m_i}\log 10}{s_{m_i}} = \log 10.$$

Since $(s_m)_{m\geq 1}$ is a subsequence of $(q_n)_{n\geq 1}$, Proposition 4.7 implies

$$\log\theta(\alpha_4) \geq -\liminf_{m\to\infty} \frac{\log\|s_m\alpha_4\|}{s_m}.$$

But we readily note

$$\|s_m\alpha_4\| = \mathcal{EXP}_m(10) \sum_{k=m+1}^{\infty} \frac{1}{\mathcal{EXP}_k(10)} \leq \frac{2\,\mathcal{EXP}_m(10)}{\mathcal{EXP}_{m+1}(10)}.$$

Thus we find

$$\begin{aligned}
\log\theta(\alpha_4) &\geq -\liminf_{m\to\infty} \frac{\log(2s_m/s_{m+1})}{s_m} \\
&= -\liminf_{m\to\infty} \frac{\log 2 + \log s_m - \log s_{m+1}}{s_m} = \limsup_{m\to\infty} \frac{s_m\log 10}{s_m} = \log 10.
\end{aligned}$$

$\square$

Let $\beta > 1$ be fixed. We define two integer sequences $(a_n)_{n\geq 0}$ and $(q_n)_{n\geq -1}$ as follows. Firstly suppose $q_{-1} = 0$, $q_0 = 1$ and a_0 is any integer. Now define a_n and q_n by

$$a_n = \left\lfloor \frac{\beta^{q_{n-1}}}{q_{n-1}} \right\rfloor, \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \geq 1.$$

One can find that these recursive equations uniquely determine a_n, q_n, and also that q_n is the denominator of the n'th convergent of $[a_0; a_1, a_2, \ldots]$. The next example tells us that possible irrationality bases exhaust all real numbers in $[1, \infty)$.

Example 7. Let a sequence $(a_n)_{n\geq 0}$ be as above and $\alpha_5 = [0; a_1, a_2, \ldots]$. Then $\theta(\alpha_5) = \beta$. So $\Delta'(\lfloor\beta-1\rfloor + \alpha_5) = 0$ but $\Delta(x)$ is not differentiable at $x = \lfloor\beta-2\rfloor + \alpha_5$.

Proof. By noting that

$$\beta^{q_n} - q_n \leq a_{n+1}q_n < q_{n+1} < (a_{n+1}+1)q_n \leq \beta^{q_n} + q_n,$$

we have

$$\limsup_{n\to\infty} \frac{\log(\beta^{q_n} - q_n)}{q_n} \leq \log\theta(\alpha_5) \leq \limsup_{n\to\infty} \frac{\log(\beta^{q_n} + q_n)}{q_n},$$

that is, $\theta(\alpha_5) = \beta$. $\qquad\square$

The following are typical examples of hyper-exponential Liouville numbers. We leave their proofs to the reader.

Example 8. Suppose that a_0 is an arbitrary integer and for $n \geq 1$ let

$$a_n = \mathcal{EXP}_n(n), \ b_n = \mathcal{EXP}_{2^n}(2^n).$$

Then $\alpha_6 = [a_0; a_1, a_2, \ldots]$ and $\alpha_7 = \sum_{k=1}^{\infty} 1/b_n$ are hyper-exponential Liouville numbers.

We have seen some examples of hypo-exponential, exponential and hyper-exponential Liouville numbers. In light of our problem, a natural question arises whether the derivative of Δ can assume a nonzero real number.

Question. Does there exist a real α such that $\theta(\alpha) = \Delta(\alpha)$? More specifically, is it possible that Δ is differentiable at α and $0 < \Delta'(\alpha) < \infty$?

References

[1] B. Adamczewski and Y. Bugeaud. On the complexity of algebraic numbers II. continued fractions. *Acta Math.* **195** (2005) 1-20.

[2] B. Adamczewski and Y. Bugeaud. On the complexity of algebraic numbers I. expansions in integer bases. *Ann. of Math. (2)* to appear.

[3] B. Adamczewski and Y. Bugeaud. Dynamics for β-shifts and Diophantine approximation. *Ergodic Theory Dynam. Systems* to appear.

[4] B. Adamczewski and J. Cassaigne. Diophantine properties of real numbers generated by finite automata. *Compos. Math.* **142** (2006) 1351-1372.

[5] J.-P. Allouche, J.L. Davison, M. Queffélec and L.Q. Zamboni. Transcendence of Sturmian or morphic continued fractions. *J. Number Theory* **91** (2001) 39-66.

[6] F. Blanchard. β-expansions and symbolic dynamics. *Theoret. Comput. Sci.* **65** (1989) 131-141.

[7] S. Bruiltet. D'une mesure d'approximation simultanée à une mesure d'irrationalité: le cas de $\Gamma(1/4)$ et $\Gamma(1/3)$. *Acta Arith.* **104** (2002) 243-281.

[8] S. Bullett and P. Sentenac. Ordered orbits of the shift, square roots, and the devil's staircase. *Math. Proc. Cambridge Philos. Soc.* **115** (1994) 451-481.

[9] D.P. Chi and D.Y. Kwon. Sturmian words, β-shifts, and transcendence. *Theoret. Comput. Sci.* **321** (2004) 395-404.

[10] C.S. Davis. Rational approximations to *e. J. Austral. Math. Soc. Ser. A* **25** (1978) 497-502.

[11] S. Ferenczi and C. Mauduit. Transcendence of numbers with a low complexity expansion. *J. Number Theory* **67** (1997) 146-161.

[12] A. Gelfond. Sur les approximations des nombres transcendants par des nombres algebriques. *C. R. (Dokl.) Acad. Sci. URSS* **2** (1935) 177-179, French text 180-182.

[13] G.H. Hardy and E.M. Wright. *An introduction to the theory of numbers.* Oxford University Press, 1979.

[14] M. Hata. Legendre type polynomials and irrationality measures. *J. Reine Angew. Math.* **407** (1990) 99-125.

[15] M. Hata. Rational approximations to π and some other numbers. *Acta Arith.* **63** (1993) 335-349.

[16] A.Ya. Khinchin. *Continued fractions.* The University of Chicago Press, 1964.

[17] D.Y. Kwon. Beta-numbers whose conjugates lie near the unit circle. *Acta Arith.* (2006) to appear.

[18] M. Lothaire. *Algebraic combinatorics on words.* Cambridge University Press, 2002.

[19] W. Parry. On the β-expansion of real numbers. *Acta Math. Acad. Sci. Hungar.* **11** (1960) 401-416.

[20] G. Rhin and C. Viola. The group structure for $\zeta(3)$. *Acta Arith.* **97** (2001) 269-293.

[21] A.M. Rockett and P. Szüsz. *Continued fractions.* World Scientific, 1992.

[22] K.F. Roth. Rational approximations to algebraic numbers. *Mathematika* **2** (1955) 1-20; corrigendum 168.

[23] J. Sondow. *An irrationality measure for Liouville numbers and conditional measures for Euler's constant.* 23rd Journées Arithmétiques, Graz, Austria (2003). See also *http://arxiv.org/abs/math.NT/0406300.*

[24] J.-L. Verger-Gaugry. On gaps in Rényi β-expansions of unity for $\beta > 1$ an algebraic number. *Ann. Inst. Fourier (Grenoble)* (2006) to appear.

[25] M. Waldschmidt. *Introduction to recent results in transcendental number theory.* Workshop and conference in number theory, Hong-Kong (1993) MSRI Preprint #1993-074.

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